Electrostatic Potential Penetration into Two Circular Apertures in a Thick Conducting Plane

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A boundary-value problem of electrostatic potential penetration into two circular apertures in a thick conducting plane is solved rigorously. The scattered potentials are represented in terms of discrete and continuous modes by using the Hankel transform and superposition principle. The boundary conditions are enforced to obtain a set of simultaneous equations for the discrete modal coefficients. The electric polarizability is represented in fast convergent series which is amenable to computation. The electric polarizability is evaluated for various aperture geometries to show the characteristics of electrostatic potential penetration.

Keywords circular aperture, polarizability, Hankel transform

Introduction

An electromagnetic coupling between circular apertures in a conducting plane has been studied in (Savov, 1993; Bird, 1996; Bailey & Bostian, 1974) due to its applications in frequency selective surface (FSS) and electromagnetic interference and compatibility. Electrostatic and magnetostatic potential penetration into circular apertures in conducting planes have been also investigated for various application purposes (Okon & Harrington, 1981; Gluckstern & Diamond, 1991; Lee & Eom, 1996; Sten, 1999a, 1999b, Eom et al., 2001). It is of theoretical interest to analyze the behavior of electrostatic potential penetration into multiple circular apertures in a thick conducting plane, but its investigation is very few. In this paper, we will investigate an electrostatic potential penetration into two circular apertures in a thick conducting plane. The present paper is a continuation of (Eom et al., 2001), where a problem of magnetostatic potential penetration into two circular apertures in a thick conducting plane was considered. We will obtain a rigorous solution for the boundary-value problem dealing with two circular apertures in a thick conducting plane by using the Hankel transform, the superposition principle, and mode-matching.
Potential Representations

Consider a primary electrostatic potential impinging on two circular apertures in a thick conducting plane as shown in Figure 1. In region (I) \((z > 0)\), the total electrostatic potential consists of the incident and scattered components. It is convenient to represent the scattered potential based on the superposition principle.

\[
\Phi^i(z) = z,
\]

\[
\Phi^s(r, \phi, z) = \Phi_{s1}(r, \phi, z) + \Phi_{s2}(r', \phi', z),
\]

where

\[
\Phi_{s1}(r, \phi, z) = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_{0}^{\infty} \Phi_{s1}^m(\zeta) J_m(\zeta r)e^{-\zeta z} \zeta d\zeta,
\]

\[
\Phi_{s2}(r', \phi', z) = \sum_{m=-\infty}^{\infty} e^{im\phi'} \int_{0}^{\infty} \Phi_{s2}^m(\zeta) J_m(\zeta r')e^{-\zeta z} \zeta d\zeta.
\]

In region (II) \((r < a_1, -d < z < 0)\), the electrostatic potential takes the form of

\[
\Phi^{II}(r, \phi, z) = \sum_{m=-\infty}^{\infty} e^{im\phi} \sum_{n=1}^{\infty} \left[ A_{mn} \sinh k_{mn}(z + d) + B_{mn} \cosh k_{mn}(z + d) \right] J_m(k_{mn}r),
\]

where \(k_{mn}\) is determined by \(J_m(k_{mn}a_1) = 0\). In region (III) \((r' < a_2, -d < z < 0)\), the electrostatic potential is

\[
\Phi^{III}(r', \phi', z) = \sum_{m=-\infty}^{\infty} e^{im\phi'} \sum_{n=1}^{\infty} \left[ C_{mn} \sinh t_{mn}(z + d) + D_{mn} \cosh t_{mn}(z + d) \right] J_m(t_{mn}r'),
\]

\[\text{Figure 1. Problem geometry.}\]
where \( t_{mn} \) is determined by \( J_m(t_{mn}a_2) = 0 \). In region (IV) \((z < -d)\), the transmitted potential is also represented based on the superposition principle.

\[
\Phi^t = \Phi_{t1}(r, \phi, z) + \Phi_{t2}(r', \phi', z),
\]

where

\[
\Phi_{t1}(r, \phi, z) = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{\infty} \Phi_{t1}^m(\zeta) J_m(\zeta r) e^{\zeta(z+d)} \zeta d\zeta,
\]

\[
\Phi_{t2}(r', \phi', z) = \sum_{m=-\infty}^{\infty} e^{im\phi'} \int_0^{\infty} \Phi_{t2}^m(\zeta) J_m(\zeta r') e^{\zeta(z+d)} \zeta d\zeta.
\]

### Enforcement of Boundary Conditions

We enforce the boundary conditions to solve equations for the modal coefficients \( A_{mn}, B_{mn}, C_{mn} \), and \( D_{mn} \). The continuities of potentials at \( z = 0, -d \) require, respectively,

\[
\Phi^i(r, \phi, 0) + \Phi_x^i(r, \phi, 0) = \begin{cases} 
\Phi^H(r, \phi, 0), & r < a_1, \\
0, & r > a_1,
\end{cases}
\]

(10)

\[
\Phi^i(r', \phi', 0) + \Phi_x^i(r', \phi', 0) = \begin{cases} 
\Phi^H(r', \phi', 0), & r' < a_2, \\
0, & r' > a_2,
\end{cases}
\]

(11)

\[
\Phi_{t1}(r, \phi, -d) = \begin{cases} 
\Phi^H(r, \phi, -d), & r < a_1, \\
0, & r > a_1,
\end{cases}
\]

(12)

\[
\Phi_{t2}(r', \phi', -d) = \begin{cases} 
\Phi^H(r', \phi', -d), & r' < a_2, \\
0, & r' > a_2,
\end{cases}
\]

(13)

Applying the Hankel transform to (10) through (13) yields, respectively,

\[
\tilde{\Phi}_{t1}^m(\zeta) = \sum_{n=1}^{\infty} \left( A_{mn} \sinh k_{mn}d + B_{mn} \cosh k_{mn}d \right) \\
\times \left[ \frac{-a_1k_{mn} J_m(\zeta a_1) J_{m+1}(k_{mn}a_1)}{\zeta^2 - k_{mn}^2} \right],
\]

(14)

\[
\tilde{\Phi}_{t2}^m(\zeta) = \sum_{n=1}^{\infty} \left( C_{mn} \sinh t_{mn}d + D_{mn} \cosh t_{mn}d \right) \\
\times \left[ \frac{-a_2t_{mn} J_m(\zeta a_2) J_{m+1}(t_{mn}a_2)}{\zeta^2 - t_{mn}^2} \right],
\]

(15)

\[
\tilde{\Phi}_{t1}^m(\zeta) = \sum_{n=1}^{\infty} B_{mn} \left[ \frac{-a_1k_{mn} J_m(\zeta a_1) J_{m+1}(k_{mn}a_1)}{\zeta^2 - k_{mn}^2} \right],
\]

(16)

\[
\tilde{\Phi}_{t2}^m(\zeta) = \sum_{n=1}^{\infty} D_{mn} \left[ \frac{-a_2t_{mn} J_m(\zeta a_2) J_{m+1}(t_{mn}a_2)}{\zeta^2 - t_{mn}^2} \right].
\]

(17)
The continuity of potential derivative at $z = 0$ for $0 < r < a_1$,

$$\left. \frac{\partial [\Phi^i(z) + \Phi^f]}{\partial z} \right|_{z=0} = \left. \frac{\partial \Phi^{Ii}(r, \phi, z)}{\partial z} \right|_{z=0},$$  \hspace{1cm} (18)

is rewritten as

$$1 - \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^\infty \Phi^m_s(\xi) J_m(\xi r) \xi^2 d\xi - \sum_{p=-\infty}^{\infty} e^{ip\phi'} \int_0^\infty \Phi^p_s(\xi) J_p(\xi r') \xi^2 d\xi = \sum_{m=-\infty}^{\infty} (A_{mn} \cosh k_{mn} d + B_{mn} \sinh k_{mn} d) k_{mn} J_m(k_{mn} r).$$  \hspace{1cm} (19)

Graf’s addition theorem gives

$$J_p(\xi r') e^{ip\phi'} = \sum_{m=-\infty}^{\infty} J_{m-p}(\xi) J_m(\xi r) e^{im\phi}.$$  \hspace{1cm} (20)

Substituting (20) in (19), multiplying (19) by $J_m(k_{mq} r)$, and integrating with respect to $r$ from $0$ to $a_1$, we get

$$X - \sum_{n=1}^{\infty} (A_{mn} \sinh k_{mn} d + B_{mn} \cosh k_{mn} d) a_1^2 k_{mn} k_{mq} J_{m+1}(k_{mn} a_1) J_{m+1}(k_{mq} a_1) I_1$$

$$- \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} (C_{pn} \sinh t_{pn} d + D_{pn} \cosh t_{pn} d) a_1 a_2 k_{mq} t_{pn} J_{m+1}(k_{mq} a_1) J_{p+1}(t_{pn} a_2) I_2$$

$$= \frac{a_1^2}{2} k_{mq} J_{m+1}(k_{mq} a_1)^2 (A_{mq} \cosh k_{mq} d + B_{mq} \sinh k_{mq} d),$$  \hspace{1cm} (21)

where

$$X = \delta m_0 \int_0^{a_1} J_m(k_{mq} r) r dr,$$  \hspace{1cm} (22)

$$I_1 = \int_0^\infty \frac{J_m(\xi a_1)^2 \xi^2}{(\xi^2 - k_{mn}^2)(\xi^2 - k_{mq}^2)} d\xi,$$  \hspace{1cm} (23)

$$I_2 = \int_0^\infty \frac{J_m(\xi a_1) J_{m-p}(\xi l) J_p(\xi a_2) \xi^2}{(\xi^2 - t_{pn}^2)(\xi^2 - k_{mq}^2)} d\xi.$$  \hspace{1cm} (24)

Similarly, the boundary condition at $z = 0$ for $0 < r' < a_2$

$$\left. \frac{\partial [\Phi^i(z) + \Phi^f]}{\partial z} \right|_{z=0} = \left. \frac{\partial \Phi^{II}(r', \phi', z)}{\partial z} \right|_{z=0}, \hspace{1cm} 0 < r' < a_2,$$  \hspace{1cm} (25)
Potential Penetration into Circular Apertures

\[ Y = \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( A_{pn} \sinh k_{pn}d + B_{pn} \cosh k_{pn}d \right) a_1 a_2 k_{pn} J_{p+1}(k_{pn}a_1) J_{m+1}(t_{mq}a_2) I_3 \]

\[ - \sum_{n=1}^{\infty} \left( C_{mn} \sinh t_{mn}d + D_{mn} \cosh t_{mn}d \right) a_2^2 t_{mn} t_{mq} J_{m+1}(t_{mn}a_2) J_{m+1}(t_{mq}a_2) I_4 \]

\[ = \frac{a_2^2}{2} t_{mq} J_{m+1}(t_{mq}a_2)^2 \left( C_{mq} \cosh t_{mq}d + D_{mq} \sinh t_{mq}d \right), \quad (26) \]

where

\[ Y = \delta_{m0} \int_{0}^{a_2} J_m(t_{mq}r) r dr, \quad (27) \]

\[ I_3 = \int_{0}^{\infty} \frac{J_p(\xi a_1) J_{p-m}(\xi l) J_m(\xi a_2) \xi^2}{(\xi^2 - k_{pn}^2)(\xi^2 - t_{mq}^2)} d\xi, \quad (28) \]

\[ I_4 = \int_{0}^{\infty} \frac{J_m(\xi a_2)^2 \xi^2}{(\xi^2 - t_{mn}^2)(\xi^2 - t_{mq}^2)} d\xi. \quad (29) \]

The boundary conditions at \( z = -d \) for \( 0 < r < a_1 \) and \( 0 < r' < a_2 \),

\[ \frac{\partial \Phi'}{\partial z} \bigg|_{z=-d} = \frac{\partial \Phi''}{\partial z} \bigg|_{z=0}, \quad 0 < r < a_1, \quad (30) \]

\[ \frac{\partial \Phi'}{\partial z} \bigg|_{z=-d} = \frac{\partial \Phi''}{\partial z} \bigg|_{z=0}, \quad 0 < r' < a_2, \quad (31) \]

also yield, respectively,

\[ -\frac{a_1^2}{2} k_{mq} J_{m+1}(k_{mq}a_1)^2 A_{mq} + \sum_{n=1}^{\infty} B_{mn} a_1^2 k_{mn} k_{mq} J_{m+1}(k_{mn}a_1) J_{m+1}(k_{mq}a_1) I_1 \]

\[ + \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} D_{pn} a_1 a_2 k_{mq} t_{pn} J_{m+1}(k_{mn}a_1) J_{p+1}(t_{pn}a_2) I_2 = 0, \quad (32) \]

\[ \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} B_{pn} a_1 a_2 k_{pn} t_{mq} J_{p+1}(k_{pn}a_1) J_{m+1}(t_{mq}a_2) I_3 - \frac{a_2^2}{2} t_{mq} J_{m+1}(t_{mq}a_2)^2 C_{mq} \]

\[ + \sum_{n=1}^{\infty} D_{mn} a_2^2 t_{mn} t_{mq} J_{m+1}(t_{mn}a_2) J_{m+1}(t_{mq}a_2) I_4 = 0. \quad (33) \]

It is possible to solve a set of simultaneous equations (21), (26), (32), and (33) for the modal coefficients \( A_{mn}, B_{mn}, C_{mn}, \) and \( D_{mn} \).
Numerical Computations

The electric polarizability has been used in Gluckstern and Diamond (1991) and Lee and Eom (1996) to analyze the behavior of field penetration into an aperture. The electric polarizability is given by:

for \(-d < z < 0\) and \(0 < r < a_1\),

\[
\chi_1(z) \equiv 4\pi \int_0^{a_1} \Phi^I(r, z) r dr
\]

\[
= 4\pi a_1 \sum_{n=1}^{\infty} \left[ A_{0n} \sinh k_0 n (z + d) + B_{0n} \cosh k_0 n (z + d) \right] J_1(k_0 a_1)/k_0 n,
\] (34)

for \(-d < z < 0\) and \(0 < r' < a_2\),

\[
\chi_2(z) \equiv 4\pi \int_0^{a_2} \Phi^II(r', z) r' dr'
\]

\[
= 4\pi a_2 \sum_{n=1}^{\infty} \left[ C_{0n} \sinh t_0 n (z + d) + D_{0n} \cosh t_0 n (z + d) \right] J_1(t_0 a_2)/t_0 n.
\] (35)

To check the accuracy of our results, we plot the excess polarizability for a circular aperture at \(z = -d\) in Figure 2. The excess polarizability is defined as a ratio of an increase in the polarizability due to the neighboring aperture to the polarizability of a single aperture.

\[
\text{Excess polarizability} = \frac{\chi_1(z) - \chi_1(z)|_{\text{single}}}{\chi_1(z)|_{\text{single}}}.
\] (36)

Our results agree very well with Sten (1999a) when the thickness of the conducting plane is extremely small \((d/a = 0.0001)\). Figure 3 shows the excess polarizability of the aperture \(a_1\) versus the displacement \((l - a_1 - a_2)/a_1\) for the different ratios of aperture radii \((a_2/a_1)\). The number of modes used in the computations is \(m = 3\) (azimuthal direction) and \(n = 6\) (radial direction), indicating the rapid convergence of our series solution. The excess polarizability decreases as the ratio \((a_2/a_1)\) decreases. Figure 4 depicts the total excess polarizability of two apertures versus the displacement \((l - a_1 - a_2)/a_1\) for the different ratios of aperture radii \((a_2/a_1)\):

\[
\text{Total excess polarizability} = \frac{\chi_1(z) + \chi_2(z) - (\chi_1(z)|_{\text{single}} + \chi_2(z)|_{\text{single}})}{\chi_1(z)|_{\text{single}} + \chi_2(z)|_{\text{single}}}.
\] (37)

The total excess polarizability decreases as the displacement increases. It is interesting to note that the total excess polarizability becomes maximum when the radii of two apertures are the same. This implies that maximum coupling between two adjacent apertures occurs when two apertures are identical. Figure 5 illustrates the polarizability versus the thickness \((d/a)\) of a conducting plane at \(z = 0, -d\). The comparison between our results and the single aperture case (Lee & Eom, 1996) also shows favorable agreement. The polarizability of two circular apertures is almost the same as that of a single aperture irrespective of the aperture thickness when \(l/a = 3\). Note that the polarizabilities reduce slightly when the displacement \((l/a)\) increases.
Figure 2. Excess polarizability as a function of $l/a$ ($a_1 = a_2 = a, d/a = 0.00001$).

Figure 3. Excess polarizability for a single aperture as a function of $(l - a_1 - a_2)/a_1$ ($d/a_1 = 0.00001$).
Figure 4. Total excess polarizability for two apertures as a function of \((l - a_1 - a_2)/a_1\) \((d/a_1 = 0.00001)\).

Figure 5. Polarizability as a function of \(d/a\) \((a_1 = a_2 = a, \chi_1(z) = \chi_2(z) = \chi(z))\).
Conclusion

The electrostatic potential penetration into two circular apertures in a thick conducting plane is investigated. A rigorous solution is obtained by using the Hankel transform, superposition, and mode-matching method. The solution is represented by a fast-convergent series, which is numerically efficient. The effects of a displacement between two apertures and the radius of two apertures on the electric polarizability are discussed. It is found that the maximum excess polarizability occurs when two apertures are identical.

References


