Electric Polarizability of Multiple Annular Apertures in a Thick Conducting Plane

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This paper examines electrostatic potential penetration into multiple annular apertures in a thick conducting plane. A boundary-value problem of electrostatic potential at thick multiple annular apertures is solved rigorously by using the Hankel transform and superposition. The scattered potentials are represented based on eigenfunction expansion in terms of discrete and continuous modes. The boundary conditions are used to derive a set of simultaneous equations for the discrete modal coefficients. A rapidly convergent series representation for the electric polarizability is presented. The electric polarizability is calculated in terms of aperture geometry to illustrate the behavior of electrostatic potential penetration.

Keywords annular aperture, Hankel transform, electric polarizability

1. Introduction

Investigators have studied electrostatic potential penetration into apertures in a thick conducting plane due to its applications in the radiation from small holes in waveguides and resonant cavities. In order to investigate the potential penetration, electric polarizability of apertures has been derived, based on the moment method (Okon & Harrington, 1981) and on the variational method (Gluckstern & Diamond, 1991). There have been extensive studies on the electric polarizability of rectangular apertures (Park & Eom, 1998) and circular apertures (Park & Eom, 2002) in a thick conducting plane based on Fourier transform and eigenfunction expansion. Electric polarizabilities of a single concentric annular aperture (Lee & Eom, 1999) and an eccentric annular aperture (Park & Eom, 2005) also have been derived based on the Hankel transform and conformal mapping, respectively. However, the theoretical analysis of electric polarizability of multiple annular apertures in a thick conducting plane seems to be lacking. In this paper, we will investigate the electrostatic potential penetration into multiple annular apertures in a thick conducting plane. In order to understand the potential penetration, the boundary-value problem associated with thick multiple annular apertures must be solved. We will use Hankel transform, eigenfunction expansion, and superposition to derive a rigorous
solution of the boundary-value problem. The electric polarizability is calculated using our rapidly convergent series solution in terms of aperture geometry to show the behavior of electrostatic potential penetration. Notations used in this paper are somewhat similar to those in previous studies (Park & Eom, 2002, 2005).

2. Potential Representations

Consider an incident electrostatic potential impinging on \( N \) annular apertures in a thick conducting plane \( (j = 1, 2, \ldots, N) \) (see Figure 1). Assume that the thick conducting
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plane and each inner conductor are at zero potential. In region (I) \((z > 0)\), the total electrostatic potential is a sum of the incident and scattered components. The scattered potential is represented based on the Hankel transform and superposition (Park & Eom, 2002).

\[
\Phi^I = z + \sum_{j=1}^{N} \Phi^I_j(r_j, \phi_j, z)
\]

(1)

where

\[
\Phi^I_j(r_j, \phi_j, z) = \sum_{m=-\infty}^{\infty} e^{i m \phi_j} \int_{0}^{\infty} \Phi^I_{mj}(\zeta) J_m(\zeta r) e^{-\kappa \zeta} \zeta d\zeta
\]

(2)

In region (II) \((a_j < r_j < b_j, -d < z < 0)\), the electrostatic potential is given by

\[
\Phi^{II}_j(r_j, \phi_j, z) = \sum_{m=-\infty}^{\infty} e^{i m \phi_j} \sum_{n=1}^{\infty} [A_{mn}^I e^{k_{mn}^I(z+d)} + B_{mn}^I e^{-k_{mn}^I(z+d)}] R_m(k_{mn}^I r_j)
\]

(3)

\[
R_m(k_{mn}^I r_j) = J_m(k_{mn}^I r_j) - \frac{J_m(k_{mn}^I a_j)}{N_m(k_{mn}^I a_j)} N_m(k_{mn}^I r_j)
\]

(4)

where \(k_{mn}^I\) is determined by \(R_m(k_{mn}^I b_j) = 0\). In region (III) \((z < -d)\), the electrostatic potential is also expressed based on Hankel transform and superposition as

\[
\Phi^{III}_j(r_j, \phi_j, z) = \sum_{m=-\infty}^{\infty} e^{i m \phi_j} \int_{0}^{\infty} \Phi^{III}_{mj}(\zeta) J_m(\zeta r) e^{\xi(z+d)} \zeta d\zeta
\]

(5)

3. Enforcement of Boundary Conditions

We enforce the boundary conditions on the field continuities. The continuities of potentials at \(z = 0, -d\) require, respectively,

\[
\Phi^I(0) + \Phi^I_j(r_j, \phi_j, 0) = \begin{cases} 
\Phi^{III}_j(r_j, \phi_j, 0), & a_j < r_j < b_j \\
0, & r_j < a_j \text{ or } r_j > b_j
\end{cases}
\]

(7)

\[
\Phi^{III}_j(r_j, \phi_j, -d) = \begin{cases} 
\Phi^I_j(r_j, \phi_j, -d), & a_j < r_j < b_j \\
0, & r_j < a_j \text{ or } r_j > b_j
\end{cases}
\]

(8)

Applying the Hankel transform to (7) and (8) gives, respectively,

\[
\tilde{\Phi}^I_{mj}(\zeta) = \sum_{n=1}^{\infty} [A_{mn}^I e^{k_{mn}^I d} + B_{mn}^I e^{-k_{mn}^I d}] C_m^I(\zeta)
\]

(9)

\[
\tilde{\Phi}^{III}_{mj}(\zeta) = -\sum_{n=1}^{\infty} [A_{mn}^I + B_{mn}^I] C_m^I(\zeta)
\]

(10)
where

$$C_{mn}^{(j)}(\xi) = \frac{2}{\pi} \left[ \frac{J_m(\xi a_j)}{N_m(k_{mn}^j a_j)} - \frac{J_m(\xi b_j)}{N_m(k_{mn}^j b_j)} \right] \frac{1}{\xi^2 - k_{mn}^j}. \quad (11)$$

The continuity of potential derivative at \( z = 0 \) for \( a_j < r_j < b_j \),

$$\left. \frac{\partial \Phi^L(z)}{\partial z} + \Phi^L \right|_{z=0} = \left. \frac{\partial \Phi^{II}(r_j, \phi_j, z)}{\partial z} \right|_{z=0} \quad (12)$$

is rewritten as

$$1 - \sum_{j=1}^{N} \sum_{m=-\infty}^{\infty} e^{i m \phi_j} \int_{0}^{\infty} \Phi_{mj}^L(\xi) J_m(\xi r_j) \xi^2 d\xi$$

$$= \sum_{m=-\infty}^{\infty} e^{i m \phi_j} \sum_{n=1}^{\infty} [A_{mn} \xi^{k_{mn}^j d} - B_{mn} \xi^{-k_{mn}^j d}] k_{mn}^j R_m(k_{mn}^j r_j). \quad (13)$$

The Graf’s addition theorem is given by

$$J_p(\xi r_t) e^{i p \phi_t} = \sum_{m=-\infty}^{\infty} J_{m-p}(\xi r_t) J_m(\xi r_t) e^{i m \phi_j} e^{i (p-m) \phi_j}. \quad (14)$$

Substituting (14) into (13), multiplying (13) by \( e^{-i m \phi_j} \) and integrating with respect to \( \phi_j \) from 0 to 2\( \pi \), we get

$$\delta_{m0} - \int_{0}^{\infty} \Phi_{mj}^L(\xi) J_m(\xi r_j) \xi^2 d\xi$$

$$= \sum_{t \neq j} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} \Phi_{mt}^L(\xi) J_{m-p}(\xi r_t) J_m(\xi r_t) e^{i m \phi_j} e^{i (p-m) \phi_j} \xi^2 d\xi$$

$$= \sum_{n=1}^{\infty} [A_{mn} \xi^{k_{mn}^j d} - B_{mn} \xi^{-k_{mn}^j d}] k_{mn}^j R_m(k_{mn}^j r_j). \quad (15)$$

Substituting (9) into (15), multiplying (15) by \( R_m(k_{mn}^j r_j) \), and integrating with respect to \( r_j \) from \( a_j \) to \( b_j \), we obtain

$$\delta_{m0} = \frac{2}{\pi k_{mn}^j} \left[ \frac{1}{N_0(k_{mn}^j b_j)} - \frac{1}{N_0(k_{mn}^j a_j)} \right] - \sum_{n=1}^{\infty} [A_{mn} \xi^{k_{mn}^j d} + B_{mn} \xi^{-k_{mn}^j d}] I_1$$

$$- \sum_{t \neq j} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} [A_{pn} \xi^{k_{mn}^j d} + B_{pn} \xi^{-k_{mn}^j d}] I_2$$

$$= \left[ A_{mn} \xi^{k_{mn}^j d} - B_{mn} \xi^{-k_{mn}^j d} \right] \frac{2}{\pi k_{mn}^j} \left[ \frac{1}{N_0^2(k_{mn}^j b_j)} - \frac{1}{N_0^2(k_{mn}^j a_j)} \right]. \quad (16)$$
where

\[ I_1 = \int_0^\infty C_{mn}^{(j)}(\zeta)C_{mq}^{(j)}(\zeta)\zeta^2 d\zeta \] (17)

\[ I_2 = e^{i(p-m)\phi_j} \int_0^\infty C_{pn}^{(j)}(\zeta)C_{mq}^{(j)}(\zeta)I_{m-p}(\zeta I_{j})\zeta^2 d\zeta \] (18)

Similarly, the boundary condition of potential derivative at \( z = -d \) for \( a_j < r_j < b_j \) gives

\[ \sum_{n=1}^{\infty} [A_{mn}^{(j)} + B_{mn}^{(j)}]I_1 + \sum_{n=1}^{N} \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( A_{pn}^{(j)} + B_{pn}^{(j)} \right)I_2 \]

\[ = \left[ A_{mq}^{(j)} - B_{mq}^{(j)} \right] \frac{2}{\pi^2 k_{mq}^{(j)}} \left[ \frac{1}{N_m^2(k_{mq}^{(j)}b_j)} - \frac{1}{N_m^2(k_{mq}^{(j)}a_j)} \right]. \] (19)

It is possible to solve a system of the simultaneous equations (16) and (19) for the modal coefficients \( A_{mn}^{(j)} \) and \( B_{mn}^{(j)} \) \((j = 1, 2, \ldots, N)\).

4. Numerical Computations

The electric polarizability on the \( j \)th aperture, \( \chi_j(z) \) is given by (Gluckstern & Diamond, 1991)

\[ \chi_j(z) = 2 \int_0^{2\pi} \int_{a_j}^{b_j} \Phi_j^{(z)}(r_j, \phi_j, z)r_j dr_j d\phi_j \]

\[ = 4\pi \sum_{n=1}^{\infty} \left[ A_{0n}^{(j)} e^{k_{0n}^{(j)}(z+d)} + B_{0n}^{(j)} e^{-k_{0n}^{(j)}(z+d)} \right] \]

\[ \cdot \frac{2}{\pi k_{0n}^{(j)^2}} \left[ \frac{1}{N_0(k_{0n}^{(j)}b_j)} - \frac{1}{N_0(k_{0n}^{(j)}a_j)} \right]. \] (20)

To check the validity of our theoretical analysis, we plot the polarizability \( \chi_1(-d) \) for two identical annular apertures in Figure 2. The number of modes used in the computations is \( m = 3 \) (azimuthal direction) and \( n = 6 \) (radial direction), implying that our series solution is rapidly convergent. Our results agree very well with the single circular aperture case (Gluckstern & Diamond, 1991) when the radius of the inner conductor is very small \((a_j/b_j = 0.01)\) and the displacement is larger than the aperture radius \(((l_{12} - 2b_j)/b_j = 2)\). Figure 3 shows the excess polarizability of a single aperture at \( z = -d \) versus the displacement \(((l_{12} - 2b_j)/b_j)\) for the different ratios of the outer conductor and inner conductor radii \((a_j/b_j)\). The excess polarizability is defined as a ratio of an increase in the polarizability due to the neighboring apertures to the polarizability of a single aperture.

\[ \text{Excess polarizability} = \frac{\chi_j(z) - \chi_j(z)_{\text{angle}}}{\chi_j(z)_{\text{angle}}} \] (21)
Figure 2. Polarizability at $z = -d$ as a function of $d/b_j$ ($l_{12}/b_j = 4$).

Figure 3. Excess polarizability for a single aperture as a function of $(l_{12} - 2b_j)/b_j$ ($d/b_j = 0.0001$).
The excess polarizability decreases as the ratio \((a_j/b_j)\) increases because the aperture area decreases and the mutual interaction between apertures becomes weak. The comparison between our results and the case of two circular apertures (Sten, 1999) generally shows a good agreement when the radius of the inner conductor is very small \((a_1/b_1 = 0.01)\) and the conducting plane is thin \((d/b_j = 0.0001)\). Figure 4 depicts the excess polarizability of a single aperture at \(z = d\) versus the displacement \(((l_{12} - b_1 - b_2)/b_1)\) for the different ratios of aperture radii \((b_2/b_1)\). The excess polarizability decreases as the ratio \((b_2/b_1)\) decreases because the smaller neighboring aperture has an insignificant effect on the polarizability of the adjacent aperture. Figure 5 illustrates the total excess polarizability of two apertures at \(z = d\) versus the displacement \(((l_{12} - b_1 - b_2)/b_1)\) for the different ratios of aperture radii \((b_2/b_1)\):

\[
\text{Total excess polarizability} = \frac{\sum_{j=1}^{N} [\chi_j(z) - \chi_j(z)_{\text{single}}]}{N}.
\]

The total excess polarizability decreases as the displacement increases. It is seen that the total excess polarizability becomes maximum when two apertures are identical. This is because maximum coupling between two adjacent apertures occurs when the radii of two apertures are the same. Similar characteristics have been also observed in the study of electrostatic potential penetration into two circular apertures (Park & Eom, 2002). Figure 6 shows the averaged polarizability of the apertures at \(z = d\) versus the aperture thickness \((d/b_j)\) for the \(N \times N (N = 1, 2, 3, 4)\) identical aperture array.

![Figure 4](image-url)  
**Figure 4.** Excess polarizability for a single aperture as a function of \((l_{12} - b_1 - b_2)/b_1\) \((d/b_1 = 0.0001, a_1/b_1 = a_2/b_2 = 0.2)\).
Figure 5. Total excess polarizability for two aperture as a function of \( (l_{12} - b_1 - b_2)/b_1 \) (\( d/b_1 = 0.0001, a_1/b_1 = a_2/b_2 = 0.2 \)).

Figure 6. Averaged polarizability for \( N \times N \) aperture array as a function of \( d/b_j \) (\( a_j/b_j = 0.2, l_{ji}/b_j = 2.1 \)).
The apertures are placed close to each other \( l_{ij}/b_j = 2.1 \). It is interesting to note that the averaged polarizability increases as \( N \) increases due to strong mutual interaction between apertures.

5. Conclusion

A boundary-value problem for the electrostatic potential at multiple annular apertures in a thick conducting plane is solved rigorously. The Hankel transform, superposition, and eigenfunction expansion are used to represent the scattered potentials in discrete and continuous modes. A set of simultaneous equations for the discrete modal coefficients is obtained from boundary conditions. The electric polarizability is represented in a fast-convergent series which is amenable to numerical analysis. Computation is performed to illustrate the behaviors of the potential penetration in terms of aperture geometry. We have found that the bigger and closer neighboring apertures have more significant effects on the polarizabilities of adjacent apertures. Also, it is interesting to note that maximum coupling between adjacent annular apertures occurs when the apertures are the same. Our rigorous formulation is useful to estimate scattering from small annular apertures with arbitrary locations in a thick conducting plane. The results of our paper find practical applications in radiation and scattering from small holes in waveguides and cavities and electromagnetic interference and compatibility. Also, our method could be used for a full wave analysis.

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References